$$(1+\tau)\varphi_1'' - [(1+\tau)k^2(x_1) + \eta^2\tau]\varphi_1 - \eta\varphi_2' = 0$$
$$\tau\varphi_2'' - (1+\tau)\eta^2\varphi_2 + \eta\varphi_1' = 0 \tag{7.5}$$

Here $k(x_1)$ is the curvature of the cross-sectional contour. Equations (7.5) agree with the equations derived in /13/ by another method that applied the principle of additional energy directly to the three-dimensional theory of equilibrium of prestressed bodies.

The domain of applicability of this theory for the buckling of thin-walled rods is studied in /14/ in the example of a rod with circular section by making a comparison with the exact solution of the stability problem for a hollow circular cylinder in a three-dimensional formulation. It is established in /14/ that Eq.(7.5) enables the critical load to be determined fairly exactly, corresponding to the rod instability mode occurring in long shells. This buckling mode is characterized by the fact that the functions $\varphi_\alpha(x_1)$ have two sign changes on the cross-sectional contour. Equations (7.5) are used in /14/ to calculate the critical laod of a rod with a complex cross-sectional profile.

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# THE EXISTENCE OF AN OPTIMAL SOLUTION IN PROBLEMS OF DETERMINING THE SHAPE OF AN ELASTIC LINE*

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The existence of an optimal solution in the problem of strain energy minimization of maximization for an elastic rod is investigated. It is established that for any elastic line shape a unique solution exists in Timoshenko's theory for the boundary conditions under consideration, while there is a case in Kirchoff's theory for an inextensible rod when the solution is not unique. A generalized optimal control exists in the optimization problem. The case when a measurable optimal control exists is investigated. Examples of the generalized control are presented.

1. Let two points O and $x_l$ be fixed in $R^3$. Connected them by an elastic line of given length $l$ so that the elastic strain energy is extremal. For this problem the load can be considered to be both distributed $\mathbf{p}(\Gamma)$, $\mathbf{m}(\Gamma)$ (vectors of the forces and moments), and lumped

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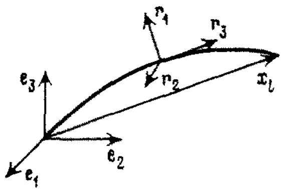


Fig.1

at the ends 0 and $l$. We let $\mathbf{e}_i$ denote the unit vectors of a fixed coordinate system, and $\mathbf{r}_i$ the unit vectors of a moving coordinate system connected to the elastic line (Fig.1). Here and henceforth everywhere, unless specified otherwise, the subscripts run through the values 1, 2, 3. Summation from 1 to 3 is assumed over the repeated subscripts in the products.

We define the rotation tensor components $\gamma$ by the relationships $\gamma_{ij} = \mathbf{e}_i^T \cdot \mathbf{r}_j$, where the superscript $T$ denotes the operation of transposition. The rotation tensor $\gamma = \gamma(\Gamma)$, where $\Gamma \in [0, l]$, determines the elastic line desired.

The equations describing the equilibrium of an elastic line can be written in the form /1/ (Timoshenko's theory)

$$P_i^{\cdot} = -p_i, \; M_i^{\cdot} = -m_i - e_{ijk}\gamma_{j3}P_k \tag{1.1}$$

$$\varphi_i^{\cdot} = A_k\gamma_{ik}\gamma_{jk}M_j, \; u^{\cdot} = -e_{ijk}\gamma_{j3}\varphi_k + B_k\gamma_{ik}\gamma_{jk}P_j$$

$$x_i^{\cdot} = \gamma_{i3}, \; \Pi^{\cdot} = \pi = {}^1/_2\,(A_k\gamma_{ik}\gamma_{jk}M_iM_j + B_k\gamma_{ik}\gamma_{jk}P_iP_j) \tag{1.2}$$

$$\left(A_k = \frac{1}{Ej_k}, \; E_k = \frac{1}{Gsi_k}\,(k = 1, 2), \; A_3 = \frac{1}{Gc}, \; B_3 = \frac{1}{Es}\right)$$

The dot denotes the derivative $d/d\Gamma$; $\mathbf{P}, \mathbf{M}, \varphi, \mathbf{u}$ are, respectively, the force, moment, angle of rotation, and displacement vectors, $\mathbf{x}$ is a vector governing the location of the elastic line, $\pi$ is the specific elastic strain energy, $e_{ijk}$ is the Levi-Civita tensor, $E$ is Young's modulus, $G$ is the shear modulus, $s$ is the area, $j_1, j_2$ are the principal moments of inertia, $i_1, i_2$ are the principal shear factors, and $c$ is the torsional stiffness of a rod section. We shall consider $p_i(\Gamma), m_i(\Gamma), \gamma_{ik}(\Gamma)$ to be measurable functions $\Gamma \in [0, l]$, where $\gamma_{ik}$ satisfy natural constraints ( $\delta_{ij}$ is the Kronecker delta)

$$\gamma_{ik}\gamma_{jk} = \delta_{ij} \tag{1.3}$$

For simplicity we consider five kinds of boundary conditions at the ends of the rod

$$\mathbf{u}(0) = \varphi(0) = \mathbf{u}(l) = \varphi(l) = 0 \tag{1.4}$$

$$\mathbf{u}(0) = \mathbf{u}(l) = \varphi(l) = 0, \quad \mathbf{M}(0) = -\mathbf{M}_0 \tag{1.5}$$

$$\varphi(0) = \mathbf{u}(l) = \varphi(l) = 0, \quad \mathbf{P}(0) = -\mathbf{P}_0 \tag{1.6}$$

$$\mathbf{u}(l) = \varphi(l) = 0, \quad \mathbf{P}(0) = -\mathbf{P}_0, \quad \mathbf{M}(0) = -\mathbf{M}_0 \tag{1.7}$$

$$\varphi(0) = \mathbf{u}(l) = 0, \quad \mathbf{P}(0) = -\mathbf{P}_0, \quad \mathbf{M}(l) = \mathbf{M}_l \tag{1.8}$$

The condition

$$x_i(0) = 0, \quad x_i(l) = x_{li}, \quad \Pi(0) = 0 \tag{1.9}$$

should be given for (1.2).

We consider as the optimization problem

$$\inf \mu\Pi(l) \tag{1.10}$$

where $\mu = 1$ or $\mu = -1$. We will call the optimization problem (1.1)–(1.3), (1.9), (1.10) with one of the boundary conditions (1.4)–(1.8), problem $A$. An analogous problem was considered in /2/ for the plane case.

In addition to problem $A$ we shall consider the split problem /3, 4/

$$P_i^{\cdot} = -p_i, \; M_i^{\cdot} = -m_i - \Sigma\lambda_t\gamma_{j3}^{(t)}e_{ijk}P_k \tag{1.11}$$

$$\varphi_i^{\cdot} = \Sigma\lambda_tA_k\gamma_{ik}^{(t)}\gamma_{jk}^{(t)}M_j, \; u_i^{\cdot} = \Sigma\lambda_t\,(-e_{ijk}\gamma_{j3}^{(t)}\varphi_k + B_k\gamma_{ik}^{(t)}\gamma_{j3}^{(t)}P_j)$$

$$x_i^{\cdot} = \Sigma\lambda_t\gamma_{i3}^{(t)}, \; \Pi^{\cdot} = \Sigma\lambda_t\pi^{(t)} \tag{1.12}$$

$$\lambda_t \geqslant 0, \; \Sigma\lambda_t = 1, \; \gamma_{ik}^{(t)}\gamma_{jk}^{(t)} = \delta_{ij}, \; t = 0, \ldots, 16 \tag{1.13}$$

Here and henceforth, $\Sigma$ denotes summation over $t$ from $t = 0$ to $t = 16$, $\lambda_t(\Gamma)$ are new control functions, $\gamma_{ik}^{(t)}(\Gamma)$ are split controls, and the expression $\pi^{(t)}$ agrees with $\pi$ (the right side of the second equation in (1.2)) in which $\gamma_{ik}^{(t)}$ are substituted in place of $\gamma_{ik}$. We assume that $\lambda_t, \gamma_{ik}^{(t)}$ are measurable functions in $[0, l]$.

We will call the optimization problem (1.11)–(1.13), (1.9), (1.10) with one of the boundary conditions (1.4)–(1.8), problem $B$.

**2.** We will investigate the existence of solutions of the boundary value problems (1.1) and (1.11) with one of the boundary conditions (1.4)–(1.8). Equations (1.11) reduce to (1.1) when $\gamma_{ik}^{(0)} = \ldots = \gamma_{ik}^{(16)}$; consequently we will investigate (1.11).

We replace the boundary value problem by its equivalent, the minimization of the additional work

$$\Pi(l) = \frac{1}{2}\langle \Sigma\lambda_t \pi^{(t)} \rangle \quad \left(\langle F \rangle \equiv \int_0^l F\, d\Gamma\right) \tag{2.1}$$

on all the $P_i, M_i$ satisfying the first two equations in (1.11) and the force and moment boundary conditions (1.4)–(1.8) as a function of the kind of fixing.

The solution of the first two equations in (1.11) can be represented in the form ($Q_i$, $N_i$ are arbitrary constants)

$$P_i = Q_i - \xi_i(\Gamma), \quad M_i = N_i - e_{ijk}x_jQ_k - \eta_i(\Gamma) \tag{2.2}$$
$$\xi_i = \int_0^\Gamma p_i\, d\Gamma, \quad \eta_i = \int_0^\Gamma (m_i - \Sigma e_{ijk}\gamma_{j3}^{(t)}\xi_k)\, d\Gamma$$

The first differential equation in (1.12) with the first condition in (1.9) was used to obtain (2.2). Substituting $P_i$ and $M_i$ from (2.2) into (2.1), we obtain

$$\Pi(l) = {}^1/_2 \langle \Sigma\lambda_t [A_k\gamma_{ik}^{(t)}\gamma_{jk}^{(t)} (N_i - e_{ijk}x_jQ_k - \eta_i)(N_j - e_{jsq}x_sQ_q - \eta_j) + B_k\gamma_{ik}^{(t)}\gamma_{jk}^{(t)} (Q_i - \xi_i)(Q_j - \xi_j)\rangle \tag{2.3}$$

Now the minimum of (2.3) should be sought for those vectors $N_i, Q_i \in U \subset R^6$, which satisfy the equations:

$Q_i = -P_{0i}$ for boundary conditions (1.6)–(1.8);

$N_i = -M_{0i}$ for boundary conditions (1.5) and (1.7);

$N_i + e_{ijk}x_{lj}Q_k - \eta_i(l) = M_{li}$ for boundary conditions (1.8).

The problem is called statically determinate for boundary conditions (1.7) and (1.8) since $N_i$ and $Q_i$ are determined by the force and moment boundary conditions (the set $U$ consists of one point).

We will now analyse $\Pi(l)$. The right side of (2.3) is a quadratic function in $N_i$ and $Q_i$, where since $\Pi(l) \geqslant 0$, $\Pi(l)$ is a convex function of $N_i$ and $Q_i$. We extract the quadratic component in $N_i, Q_i$ from (2.3), and denote it by $\Pi^\circ(N, Q)$. It follows from (1.2) that $0 < \alpha \leqslant A_k$, $0 \leqslant \beta \leqslant B_k$. Taking (1.13) into account, we obtain the estimate

$$\Pi^\circ(N, Q) \geqslant {}^1/_2\alpha\langle (N_i - e_{imn}x_mQ_n)(N_i - e_{isq}x_sQ_q)\rangle + {}^1/_2\beta l Q_iQ_i$$

and we represent it in the matrix form

$$\Pi^\circ(\mathbf{N}, \mathbf{Q}) \geqslant {}^1/_2 \| \mathbf{N}^T\mathbf{Q}^T \| \cdot D \cdot \| \mathbf{N}^T\mathbf{Q}^T \|^T \tag{2.4}$$

$$\mathbf{D} = \begin{Vmatrix} \mathbf{y} & \mathbf{w} \\ \mathbf{w}^T & \mathbf{z} \end{Vmatrix}, \quad \mathbf{w} = \| -e_{ijk}w_k \|, \quad \mathbf{z} = \| z_{ij} \|, \quad \mathbf{y} = \| \alpha l \delta_{ij} \|$$

$$w_i = \alpha\langle x_i\rangle, \; z_{ii} = \beta l + \alpha\langle x_jx_j - x_i^2\rangle \tag{2.5}$$
$$z_{ij} = -\alpha\langle x_ix_j\rangle \; (i \neq j)$$

*Theorem 1.* Let 1) $p_i$, $m_i$, $\gamma_{ij}^{(t)}$, $\lambda_t$ be measurable functions in the interval $[0, l]$ which satisfy (1.13); 2) $\alpha > 0$, $\beta > 0$. Then a unique absolutely continuous solution of (1.11) exists with one of the boundary conditions (1.4)–(1.8).

*Proof.* We will apply the theorem on minimization of a coercive functional on a convex set /5/ to (2.3). For the boundary conditions (1.4)–(1.8) the set $U$ is either a subspace or consists of one point. In order for the functional (2.3) to be coercive, it is necessary and sufficient that $\det \mathbf{D} > 0$.

We carry out three elimination steps for the elements $D_{11}, D_{22}, D_{33}$ in $\det \mathbf{D}$ by Gauss's method. Consequently we obtain

$$\det \mathbf{D} = \det \| Z_{ij} \|, \quad Z_{ii} = \alpha l z_{ii} + w_i^2 - w_jw_j$$
$$Z_{ij} = \alpha l z_{ij} + w_iw_j \; (i \neq j)$$

It follows from (2.5) that $\| Z_{ij} \|$ is a symmetric tensor of the second rank, hence we select $\mathbf{e}_i$ so that it becomes diagonal. Then $\det \mathbf{D} = Z_{11}Z_{22}Z_{33}$.

From the Cauchy inequality

$$\langle x_i\rangle^2 \leqslant l\langle x_i^2\rangle \tag{2.6}$$

it follows that

$$Z_{ii} = \alpha\beta l^2 + \alpha l\langle x_jx_j - x_i^2\rangle + \alpha^2(\langle x_i\rangle^2 - \langle x_j\rangle\langle x_j\rangle) > 0 \tag{2.7}$$

If these inequalities are satisfied, then $\det \mathrm{D} > 0$ and a unique $N_i^\circ, Q_i^\circ$ exists on which a minimum of (2.3) is reached. Substituting them into (2.2), we obtain absolutely continuous $M_i, P_i$ after which we find by substituting $M_i$ and $P_i$ into the thrid and fourth equations in (1.11) and integrating

$$\varphi_i = \varphi_{0i} + \int_0^\Gamma \Sigma\lambda_t A_k \gamma_{ik}^{(t)} \gamma_{jk}^{(t)} M_j \, d\Gamma \tag{2.8}$$

$$u_i = u_{0i} + \int_0^\Gamma \Sigma\lambda_t (-e_{ijk}\gamma_{j3}^{(t)}\varphi_k + B_k \gamma_{ik}^{(t)} \gamma_{jk}^{(t)} P_i P_j) \, d\Gamma$$

where $\varphi_{0i}, u_{0i}$ are arbitrary constants that are found from the kinematic conditions (1.4)–(1.8). It is clear that $\varphi_i, u_i$ are also absolutely continuous functions. The theorem is proved.

*Corollary 1.* Let condition 1 of Theorem 1 be satisfied, $\alpha > 0$, $B_k = 0$, and let the elastic line not lie on a straight line. Then a unique absolutely continuous solution of the boundary value problem under consideration exists.

*Proof.* We must show that $\det \mathrm{D} > 0$. The remaining discussion remains unchanged. Indeed, inequality (2.6) becomes an equality only when $x_i = \mathrm{const}$, and by virtue of (1.9) this is impossible if $x_i \equiv 0$. It follows from (2.7) that the right side of (2.7) can vanish for a certain subscript $i$ if two components of the vector $x$ are identically zero, for instance, $x_1 = x_2 \equiv 0$ (here the tensor $\| Z_{ij} \|$ has diagonal form). But for $x_1 = x_2 \equiv 0$, the rod lies on a straight line, which contradicts condition 2, therefore, $\det \mathrm{D} > 0$.

*Corollary 2.* Let condition 1 of Theorem 1 be satisfied, $\alpha > 0$, $B_k = 0$ and let the elastic line be straight. Then an absolutely continuous solution of the boundary value problem exits for the boundary conditions under consideration which is unique for boundary conditions (1.6)–(1.8), and not unique for boundary conditions (1.4) and (1.5).

*Proof.* If we select $\mathrm{r}_3 = \pm \mathrm{e}_3$, it can be seen that $\Pi(l)$ (2.3) is independent of $Q_3$. In this case

$$w_1 = w_2 = z_{33} = z_{ij} = 0\ (i \neq j), \quad w_3 = \alpha \int_0^\Gamma x_3 \, d\Gamma, \ \ z_{11} = z_{22} = \alpha \int_0^\Gamma x_3^2 \, d\Gamma$$

and (2.4) becomes the inequality $\Pi^\circ(\mathrm{N}, \mathrm{Q}) \geq {}^1/_2 \| \mathrm{N}^T Q_1 Q_2 \| \cdot \mathrm{D}^\circ \cdot \| \mathrm{N}^T Q_1 Q_2 \|^T$.

It can be seen by a direct calculation that

$$\det \mathrm{D}^\circ = \alpha l \, (l \langle x_3^2\rangle - \langle x_3\rangle^2)^2.$$

Since $x_3 \not\equiv 0$, then $\det \mathrm{D}^\circ > 0$ and therefore, unique $N_i^\circ$, $Q_1^\circ$, $Q_2^\circ$ exist which impart a minimum to $\Pi(l)$ (2.3). The constant $Q_3^\circ$ for boundary conditions (1.4) and (1.5) can be selected arbitrarily. The subsequent discussion remains the same as in Theorem 1.

3. We now consider the question of the existence of a solution for the optimal problem $A$ and $B$.

For problem $A$ we introduce the set

$$\Omega = \{y \in R^{16} \mid y_i = -p_i, \ y_{3+i} = -m_i - e_{ijk}\gamma_{j3}P_k, \tag{3.1}$$
$$y_{6+i} = A_k \gamma_{ik}\gamma_{jk} M_j, \ y_{9-i} = -e_{ijk}\gamma_{j3}\varphi_k + B_k \gamma_{ik}\gamma_{jk} P_j,$$
$$y_{12+i} = \gamma_{i3}, \ y_{16} = \pi, \ \gamma_{ik}\gamma_{jk} = \delta_{ij}\}$$

The sufficient conditions for a measurable optimal control in the problem $A$ to exist are related to the convexity of the set $\Omega$ /6/. The set $\Omega$ is not convex since even the set of controls is not convex. For problem $B$ the set analogous to (3.1) agrees with the convex hull $\Omega$ (co $\Omega$). In this connection, the existence of a solution can be shown only for problem $B$.

*Theorem 2.* Let $p_i, m_i$ be continuous functions in the interval $[0, l]$, then a measurable optimal control $\lambda_t^\circ(\Gamma)$, $\gamma_{ik}^{(t)0}(\Gamma)$, $t = 0, \ldots, 16$ exists in problem $B$.

*Proof.* Since $A_k \leq C_1, B_k \leq C_1$, we obtain the estimates

$$|P_i| \leq C_2, \ |M_i| \leq C_2, \ |\varphi_i| \leq C_2, \ |u_i| \leq C_2, \ \Pi(l) \leq C_3 \tag{3.2}$$

from Theorem 1, Corollaries 1 and 2 (for boundary conditions (1.4) and (1.5) $Q_3^\circ$ can be set equal to zero) and relationships (2.2) and (2.8), where the constants $C_i$ are independent of the controls $\lambda_t$, $\gamma_{ik}^{(t)}$, $t = 0, \ldots, 16$. Therefore, the set co $\Omega$ is convex, bounded, and continuous for any $\Gamma \in [0, l]$. The conditions for the theorem on the existence of a measurable optimal control /6/ are satisfied, and therefore the theorem is indeed valid.

The optimal control $\lambda_t^\circ, \gamma_{ik}^{(t)\circ}, t = 0, \ldots, 16$ is called a generalized optimal control (sliding mode). When $\gamma_{ik}^{(0)\circ} = \ldots = \gamma_{ik}^{(16)\circ}$, the generalized optimal control is the ordinary optimal control.

For problem $A$ the Hamiltonian reduces to the form

$$H(\gamma) = \mu(u_i p_i + \varphi_i m_i) + \mu\pi + \mu\gamma_{i3} e_{ijk}\varphi_j P_k + \rho_i\gamma_{i3} \tag{3.3}$$

where $\rho_i$ are constant Lagrange multipliers. For problem $B$ the Hamiltonian is

$$H^\circ(\lambda, \gamma) = \Sigma\lambda_t H(\gamma^{(t)}) \tag{3.4}$$

where the Hamiltonian $H(\gamma^{(t)})$ is defined by expression (3.3), in which $\gamma^{(t)}$ is substituted in palce of $\gamma$. Pontryagin's maximum principle for problem $B$ is to seek $\lambda^\circ, \gamma^{(t)\circ}$ for which

$$H^\circ(\lambda^\circ, \gamma^\circ) = \sup_{\lambda_t \geqslant 0,\ \lambda_0 + \ldots + \lambda_{16} = 1,\ \gamma_{ik}^{(t)}\gamma_{jk}^{(t)} = \delta_{ij}} H^\circ(\lambda, \gamma)$$

from which, taking into account $\lambda_t \geqslant 0$, we obtain

$$H(\gamma^{(t)0}) = \sup_{\gamma_{ik}^{(t)}\gamma_{jk}^{(t)} = \delta_{ij}} H(\gamma^{(t)}) \tag{3.5}$$

It follows from (3.5) that if the value of $\gamma^{(t)0}$ is unique for all $0 \leqslant \Gamma \leqslant l$ (with the exception, perhaps, of a finite number of points $\Gamma_n$) then the generalized optimal control is the ordinary control.

4. Let us examine the Kirchhoff inextensible rod. In this case $B_k = 0$. We use the notation $R_i = \rho_i + \mu e_{ijk}\varphi_j P_k$. To analyse condition (3.5), we take an arbitrary point $\Gamma \in [0, l]$. We combine $\mathbf{e}_3$ with $\mathbf{M}$ or $-\mathbf{M}$, and place $\mathbf{R}$ in the $\mathbf{e}_2$, $\mathbf{e}_3$ plane so that $R_2 \geqslant 0$, $R_3 \geqslant 0$ (Fig.2). Then

$$H = {}^1/_2\mu M_i M_i A_j \gamma_{3j}^2 + R_i\gamma_{i3} \tag{4.1}$$

It is clear that the unit vector $\mathbf{r}_3$ should lie in the $\mathbf{e}_2, \mathbf{e}_3$ plane between the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$. Indeed, if this were not so, then by rotating the trihedron $\mathbf{r}_j$ about $\mathbf{e}_3$ and placing $\mathbf{r}_3$ between $\mathbf{e}_2$ and $\mathbf{e}_3$ we will obtain a larger value of $H$ than prior to the rotation.

We introduce the vector $\mathbf{e}$ (Fig.2) and use the notation

$$\mathbf{r}_3\cdot\mathbf{e}_3 = \cos\theta,\ \mathbf{e}\cdot\mathbf{e}_3 = \sin\theta,\ \mathbf{r}_1\cdot\mathbf{e} = \cos\chi,\quad \mathbf{r}_2\cdot\mathbf{e} = \sin\chi$$

We then obtain from (4.1)

$$H = {}^1/_2\mu M_i M_i (A_1\cos^2\chi\sin^2\theta + A_2\sin^2\chi\sin^2\theta + A_3\cos\theta + R_2\sin\theta + R_3\cos\theta),\quad 0 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant \chi \leqslant \pi/2 \tag{4.2}$$

We will seek the maximum of the function $H$ as a function of $\chi$ and $\theta$. It follows from the condition $\partial H/\partial\chi = 0$ that a maximum of $H$ is always reached at the point $\chi = 0$ or $\chi = \pi/2$. Substituting these values into (4.2), we find

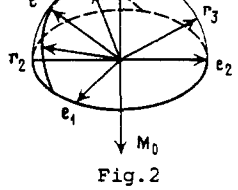

Fig.2

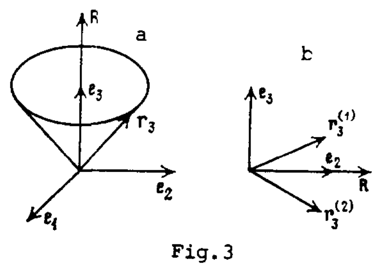

Fig.3

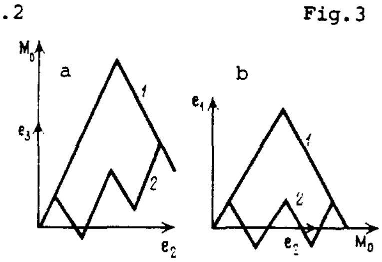

Fig.4

$$H = {}^1/_2\mu A_0 M_i M_i + {}^1/_2\mu (A_3 - A_0) M_i M_i \cos^2\theta + R_3 \cos\theta + R_2 \sin\theta, \quad A_0 = \begin{cases} A_1, & \mu = -1 \\ A_2, & \mu = 1 \end{cases} \tag{4.3}$$

From (1.2) it follows that if $E \geqslant 2G$, then $A_3 > A_1, A_3 > A_2$.

Calculating the derivatives $\partial H/\partial\theta$, $\partial^2 H/\partial\theta^2$, we conclude that a maximum of $H$ is always reached at one point for all cases except a) $\mu = 1, R_3 = 0$ b) $\mu = -1, R_3 = 0$. Therefore, the solutions of the split and initial problems agree if there are no sections on the optimal rod on which conditions a) or b) are satisfied. For case a) an entire cone of values of $\mathbf{r}_3$ exists on which $H$ reaches a maximum (Fig.3a). For case b) two values of the vector $\mathbf{r}_3$ exist on which $H$ reaches a maximum (Fig.3b).

*Example.* Let us consider the case when $B_k = 0$, $p_i = m_i = 0$ and boundary conditions (1.7) for $\mathbf{P}_0 = 0$.

The optimal rod has the form 1 (Fig.4a) in the minimization problem $\Pi(l)$. Besides this solution, there is also a generalized solution 2 (Fig.4a). In addition to solutions with breaks, a smooth optimal solution can also be constructed. Condition a is realized for these optimal solutions.

The optimal rod has the form 1 (Fig.4b) for the maximization problem $\Pi(l)$. In addition to this solution, there is also the generalized solution 2 (Fig.4b). Condition b is realized for these optimal solutions.

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# ON SYMMETRIC AND NON-SYMMETRIC CONTACT PROBLEMS OF THE THEORY OF ELASTICITY*

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Contact problems of the theory of elasticity can be subdivided into two major classes: symmetric contact problems for which the kernel of integral equations of the convolution type are even or odd functions, and non-symmetric contact problems for which the kernels are given by the sum of odd and even functions, Certain problems from this latter class were apparently examined first in /1—3/. In this paper a general approach to their study is given and an approximate solution is constructed; the results are demonstrated in two new problems.

1. As is well-known /4—6/, may plane and axisymmetric contact problems of the theory of elasticity reduce to determining the contact forces from an integral equation of the first kind with a different kernel of the form

$$\int_{-1}^{1} \varphi(\xi)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x) \quad (|x| \leqslant 1) \tag{1.1}$$

$$k(t) = \frac{1}{2} \int_{-\infty+ic}^{\infty+ic} K(\zeta)\, e^{i\zeta t}\, d\zeta \tag{1.2}$$

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